Qualifying Exam (May 2025): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1, 2, 3.

Do 2 out of problems 4, 5, 6.

Do 3 out of problems 7, 8, 9, 10, 11, 12, 13, 14

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

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(1). Consider the assignment problem shown in the following table (rows: tasks; columns: individuals; entries: costs):

	Individual		
Task	1	2	3
1	17	18	16
2	14	19	17
3	15	19	18

(a) Write the linear programming formulation of this problem.

(b) Write the dual of the linear program formulated in part (a). List the complementary slackness conditions.

(c) Using the Hungarian algorithm, find an optimal primal and an optimal dual solution.

(d) Show that the optimal primal and dual solutions found in part (c) satisfy primal feasibility, dual feasibility and complementary slackness.

(2). Consider the following integer programming problem:

$$\begin{array}{lcrcl} \max\ z & = & x_1 + 4x_2 & & \\ \text{s.t.} & & x_1 + 2x_2 & \leq & 7 \\ & & -x_1 + 2x_2 & \leq & 3 \\ & & x_1\ ,\ x_2 & \geq & 0\ ;\ x_1\ ,\ x_2 \text{ integer} \end{array}$$

Solve this problem by first finding the LP optimum and then finding the integer optimum by the cutting plane method.

(3). Consider the following LP and its optimal tableau

$$\begin{array}{rc} \min z = \mathbf{c}^T\mathbf{x} = & 50x_1 + 100x_2 \\ \text{st} & 7x_1 + 2x_2 \geq 28 \\ A\mathbf{x} \geq \mathbf{b} & 2x_1 + 12x_2 \geq 24 \\ \mathbf{x} \geq 0 & x_1, x_2 \geq 0 \end{array}$$

z	x_1	x_2	e_1	e_2	a_1	a_2	RHS
1	0	0	-5	-7.5	5-M	7.5-M	320
0	1	0	-12/80	2/80	12/80	-2/80	18/5
0	0	1	2/80	-7/80	-2/80	7/80	7/5

Here e_i are excess variables subtracted from the first and second constraints, and a_i is the artificial variable of the i-th constraint. Answer the following questions by using the tableau. Do not solve LP again!

(a) Find the dual of this LP and its optimal solution (the objective value and the values of the dual variables).

(b) Find the range of values of c_1 (currently = 50) for which the current basis remains optimal.

(c) Find the range of values of b_1 (currently $= 28$) for which the current basis remains optimal. If the new value is $b_1 = 40$, what would be the new optimal solution and new optimum?

(d) Suppose a new activity is added with cost $c_3 = 110$ and activity vector $\mathbf{a}_3 = [12, 7]^T$. Is this activity worthwhile?

(4). Consider a Poisson process $\{N(t),\ t \geq 0\}$ with arrival rate λ. Let $\{A(t)\}$ be its associated age process, i.e., $A(t) = t - S_{N(t)}$, where S_n is the time of the nth event. For a given constant $s > 0$, let $T = \inf\{t : A(t) \geq s\}$. Find $E[T]$.

(5). A Geiger counter is a device that records the arrivals of radioactive particles. Suppose the particles arrive according to a Poisson process with rate λ, and that the counter is locked for a dead period of fixed length T after each detected arrival. Let $N(t)$ be the number of detected particles during $[0, t]$. Assuming that a dead period begins at time 0, compute the probability $P(N(t) \geq k)$ for $t > kT$.

(6). Consider a single server queueing system that serves two types of customers $i = 1, 2$. Customers of type i arrive according to a Poisson process with rate λ_i, $i = 1, 2$. Assume that the arrival processes of these two types of customers are independent. The service times of successive customers are i.i.d. exponentially distributed with parameter μ. Suppose that type 1 customers have absolute priority over type 2 customers, meaning that when a type 2 customer is in service and a type 1 customer arrives, the type 2 service is interrupted and the server proceeds with the type 1 customer. Once there are no more type 1 customers in the system, the server resumes the service of the type 2 customer at the point where it was interrupted. Let L_i $(i = 1, 2)$ be the long-run average number of type i customers in system. Assuming that the system is stable, compute L_i for $i = 1, 2$.

(7). Let U be a uniform random variable on $[0, 1]$. (a) Find the density function of $1 - U^2$; (b) Use the result you obtained in part (a) in an acceptance-rejection method to generate random variates from the density function $f(x) = \frac{2}{\pi\sqrt{1-x^2}}$, $0 < x < 1$ (give the detailed steps of your algorithm).

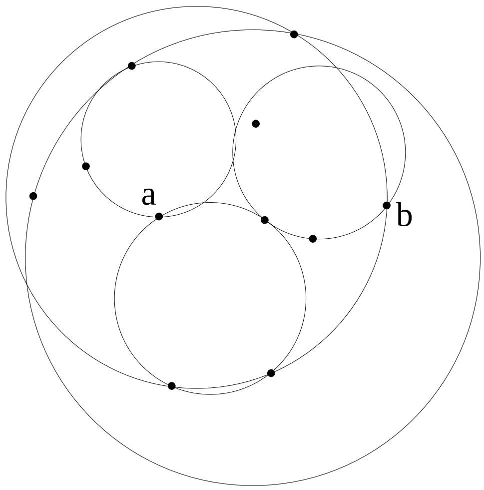
(8). Consider the stratified sampling method for estimating the integral $\int_0^1 g(x)dx$. Let $u_1, \ldots, u_n$ be n independent random numbers and define $\bar{u}_i = (u_i + i - 1)/n$ for $i = 1, \ldots, n$. Show that the stratified sampling estimator $\sum_{i=1}^n g(\bar{u}_i)/n$ reduces the variance of the crude Monte Carlo estimator, i.e.,

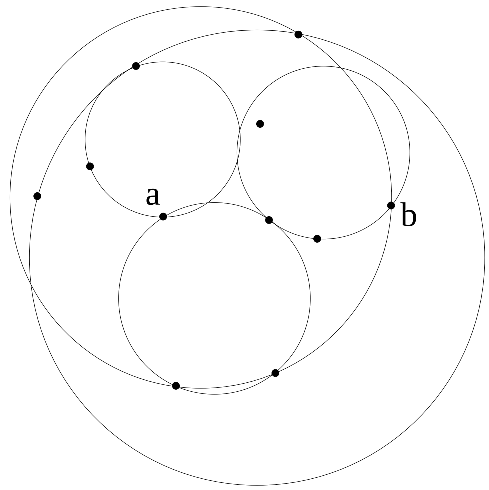
$$Var\Big(\sum_{i=1}^{n} g(\bar{u}_i)/n\Big) \leq Var\Big(\sum_{i=1}^{n} g(u_i)/n\Big).$$

(9). (a). Given a set S of n distinct points in the plane, what is the maximum/minimum number of edges in the (Euclidean) Delaunay diagram of S? How efficiently (in big-Oh notation) can the (Euclidean) Delaunay diagram be constructed for n points S in the plane? *Describe briefly (without giving details) one method that achieves the time bound in 2D.*

(b). State an important property of the Euclidean Delaunay diagram in 2D that might be used in an application. Sketch (briefly) how that property would be proved.

(c). For the set S of 11 points shown below, draw the (Euclidean) Delaunay diagram. In order to assist you in making some decisions, I have drawn some possibly relevant circles. NOTE: I provide 2 copies of the figure, in case you make a mistake on one.



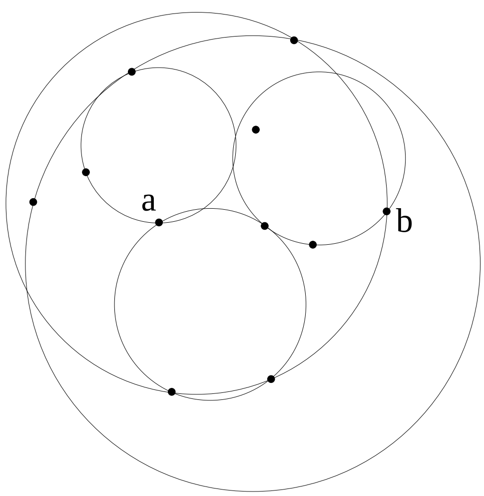


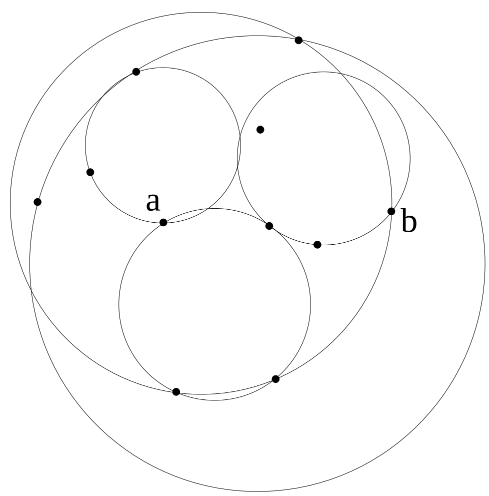
(d). Consider now the furthest-site Delaunay diagram of S.

(i). How efficiently can it be computed (in big-Oh)?

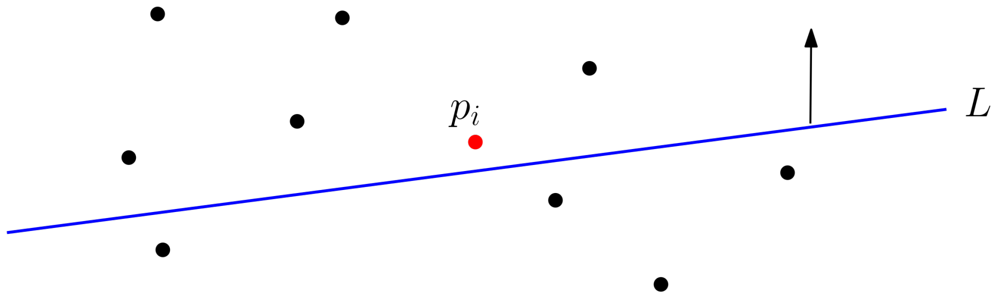
(ii). Explain briefly the relationship between the furthest-site Delaunay diagram and the three-dimensional convex hull of points related to S.

(iii). Below, draw the furthest-site Voronoi cell associated with point b.





(10). Given a set $S = \{p_1, \ldots, p_n\}$ of n points in the plane. We are to build a data structure to support efficient queries: For a (nonvertical) query line L, find the first point of S to be hit when L is translated upwards (in $+y$ direction). (or report that no point is hit, if all of S lies below L) For example, in the figure below, the result of a query with line L is to report the point p_i, shown in red.



Our goal is to have particularly efficient query time, after doing some preprocessing to construct a data structure (of reasonable size).

(a). What preprocessing/storage/query can be achieved? (Best that you know how to achieve.)

(b). Describe what steps are needed to achieve your answer to part (a). Justify your answer and describe briefly any data structures used.

(c). Now suppose we also wish to know, for the query line L, by what distance it is translated upwards before it hits a point of S. What can you say about this problem?

(11). Let X_j be i.i.d. with finite variances and zero means. Let $S_n = \sum_{j=1}^{n} X_j$. Show that $\frac{1}{n}S_n$ tends to 0 in both L^2 and in probability.

(12). Let $M = (M_n)_{n\geq 0}$ be a martingale with $M_n \in L^2$, for each n. Let S, T be bounded stopping times with $S \leq T$. Show that M_S, M_T, are both in L^2, and show that

$$E\{(M_T - M_S)^2 | \mathcal{F}_S\} = E\{M_T^2 - M_S^2 | \mathcal{F}_S\},$$

and that

$$E\{(M_T - M_S)^2\} = E\{M_T^2\} - E\{M_S^2\}.$$

(13). Consider an infinite-horizon negative dynamic programming problem with a countable state space and with finite action sets. In other words, consider a Markov Decision Process (MDP) $\{X, A, A(\cdot), p, r\}$ satisfying the following assumptions: the state space X is countable, each set $A(x)$ of available actions at a state x is finite, and all one-step rewards $r(x, a)$ are nonpositive, where $x \in X$ and $a \in A(x)$. The goal is to maximize expected total rewards. Does an optimal policy exist for this MDP? Prove your answer.

(14). Consider an infinite-horizon positive dynamic programming problem with a finite state space, compact action sets, continuous transition probabilities, and continuous one-step rewards. In other words, consider a Markov Decision Process (MDP) $\{X, A, A(\cdot), p, r\}$ satisfying the following assumptions: (i) the state space X is finite, (ii) the set of actions A is a metric space, (iii) for each state $x \in X$ the set $A(x)$ of actions available at x is compact, (iv) for every pair of states $x, y \in X$, transition probabilities $p(y|x, a)$ are continuous in $a \in A(x)$, and (v) for each $x \in X$ one-step rewards $r(x, a)$ are nonnegative and continuous functions in $a \in A(x)$. The goal is to maximize expected total rewards, which are assumed to be finite for all initial states and for all policies. Does an optimal policy exist for this MDP? Prove your answer.