Mathematical Statistics Qualifying Examination
Part I of the STAT AREA EXAM
May 27, 2025; 9:00 AM - 11:00 AM

There are 4 problems. You are required to solve them all. Show detailed work for full credit.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

NAME: ______________________________ **ID:** ______________________________

Signature: ______________________________

1. Let $X_1, \cdots, X_n$ be iid $N(\theta, \theta^2)$, $\theta > 0$. For this model both $\overline{X}$ and cS (S^2 is the sample variance) are unbiased estimators of θ, where $c = \dfrac{\sqrt{n-1}\,\Gamma[(n-1)/2]}{\sqrt{2}\,\Gamma(n/2)}$.

   (a) Prove that for any number a, $a\overline{X} + (1-a)cS$ is an unbiased estimator of θ.

   (b) Find the value of a that produces the estimator with minimum variance.

   (c) Show that $\left(\overline{X}, S^2\right)$ is a sufficient statistic for θ but it is not a complete sufficient statistic.

2. Let $X_1, \cdots, X_{10}$ be a random sample from Poisson(θ).

   (a) Find a most powerful test for H_0: $\theta = 0.1$ versus H_1: $\theta = 0.5$.

   (b) For a rejection region $\sum_{i=1}^{10} X_i \geq 3$, determine the significance level.

   (c) Find the power of the above test at $\theta = 0.5$.

3. Consider a Poisson process $\{N(t),\ t \geq 0\}$ with arrival rate λ. Let $\{A(t)\}$ be its associated age process, i.e., $A(t) = t - S_{N(t)}$, where S_n is the time of the nth event. For a given constant $s > 0$, let $T = \inf\{t : A(t) \geq s\}$. Find $E[T]$.

4. Consider a single server queueing system that serves two types of customers $i = 1, 2$. Customers of type i arrive according to a Poisson process with rate λ_i, $i = 1, 2$. Assume that the arrival processes of these two types of customers are independent. The service times of successive customers are i.i.d. exponentially distributed with parameter μ. Suppose that type 1 customers have absolute priority over type 2 customers, meaning that when a type 2 customer is in service and a type 1 customer arrives, the type 2 service is interrupted and the server proceeds with the type 1 customer. Once there are no more type 1 customers in the system, the server resumes the service of the type 2 customer at the point where it was interrupted. Let L_i ($i = 1, 2$) be the long-run average number of type i customers in system. Assuming that the system is stable, compute L_i for $i = 1, 2$.